Structural Mechanics

INTERSECTII

INTERSECTIONS

http://www.ce.tuiasi.ro/intersections

Effect of thickness variation upon plates subjected to bending

Mihai Vrabie[1], Cornel Marțincu[2], Nicolae Ungureanu[1] and Mihaela Ibănescu[1]

[1]*Faculty of Civil Engineering, Technical University "Gh. Asaki", Iasy, 700050, Romania*
[2]*S.C. IPTANA S.A., Bucharest, Romania*

Summary

The variation of the plate thickness is an element that introduces a major difficulty in solving the differential equations of the deformed middle surface, because the plate rigidities are also variable.

In the paper, the effect of the thickness variation upon shear forces is presented. These shear forces must be corrected by adding the shear stresses caused by bending and twisting moments.

A physical interpretation is given to these corrections and the separation of effects is also pointed out. There are also discussed the additional energy effects caused by thickness variation.

KEYWORDS: Bending plates, variable thickness, effect of transverse shear deformation, strain energy.

ISSN 1582-3024

INTERSECTII
INTERSECTIONS
http://www.ce.tuiasi.ro/intersections

M. Vrabie, C. Marțincu, N. Ungureanu, M. Ibănescu

1. INTRODUCTION

The plates with variable thickness are met in construction practice at diaphragms, mats, piers, settlers, dams, cantilevered roofs etc.

The variation of the plate thickness introduces major difficulties in solving the differential equations of the deformed middle surface, as the plate rigidity in bending, *D,* is variable with the plate thickness, *h*. In case of an unsymmetrical thickness variation, following different equations for extrados and intrados, the difficulties in solving the problem amplify. That is why the existent solutions in the specialized literature are referring only to particular cases for plate rigidity, supports and loading. In [1] the case of bending rigidity variable along a single direction (*D* is a linear function in terms of y) is approached. The case was analytical studied by using trigonometric series for the rectangular plate simply supported along the whole contour. For the same case, in [2] there are mentioned solutions of the problem by using the finite differences method for the differential equation of the deformed middle surface.

In [2] it is also discussed a solution obtained by substituting the actual thickness (with linear variation along a single direction) by 10 step of constant thickness and then expressing the displacement continuity conditions on the contact boundary.

The more recent approaches in this field are based on the finite element method and on the boundary element method [3], [4], [5], [6], [7].

The present paper presents the effect of thickness variation upon shear forces. A physical interpretation of shear force corrections due to thickness variation is also found.

2. THE DIFFERENTIAL EQUATION OF THE DEFORMED MIDDLE SURFACE

All the assumptions adopted in the theory of thin plates of constant thickness are valid. The rotations φ_x and φ_y of the normal lines to the middle plane are equal to the variations of the deformed middle surface slopes in the considered direction, due to Kirchhoff's assumption (Figure 1).

A continuous and slow thickness variation is admitted, so that the relations for bending and twisting moments valid for plates of constant thickness could be applied in this case with enough accuracy [1], [2], [3], so:

Structural Mechanics

INTERSECTII
http://www.ce.tuiasi.ro/intersections

$$M_x = -D\left(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right)$$
$$M_y = -D\left(\frac{\partial^2 w}{\partial y^2} + \nu\frac{\partial^2 w}{\partial x^2}\right) \qquad (1)$$
$$M_{xy} = M_{yx} = M_t = -D(1-\nu)\frac{\partial^2 w}{\partial x \partial y}$$

where the bending rigidity of the plate is variable

$$D = \frac{Eh^3(x,y)}{12(1-\nu^2)} \qquad (2)$$

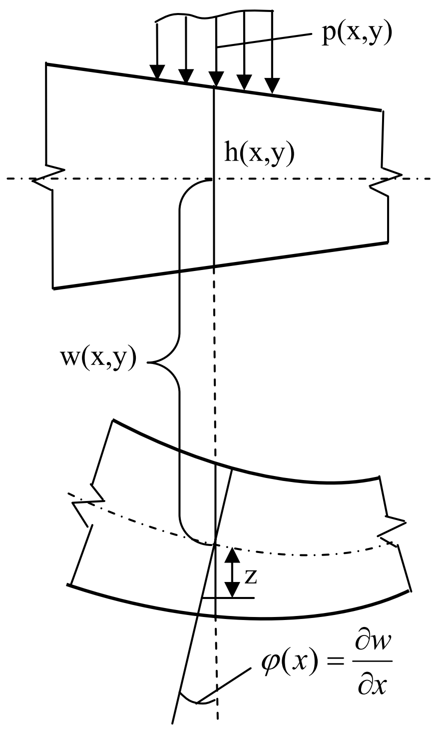

Fig. 1. A deformed segment of the bending plate

The differential equilibrium equations remain under the well-known form:

$$\begin{aligned}
&\frac{\partial Q_x}{\partial x}+\frac{\partial Q_y}{\partial y}=-p(x,y) && a)\\
&\frac{\partial M_x}{\partial x}+\frac{\partial M_{yx}}{\partial y}=Q_x && b)\\
&\frac{\partial M_{xy}}{\partial x}+\frac{\partial M_y}{\partial y}=Q_y && c)
\end{aligned}\tag{3}$$

The equation (3a) is differentiate with respect to x and the equation (3c) is differentiate with respect to y. By adding the two obtained equations and taking into account (3a), it results:

$$\frac{\partial^2 M_x}{\partial x^2}+2\frac{\partial^2 M_t}{\partial x\partial y}+\frac{\partial^2 M_y}{\partial y^2}=-p(x,y)\tag{4}$$

The moments M_x, M_y, M_t are substituted by their expressions given by (1) [3]:

$$\begin{aligned}
&-\frac{\partial^2}{\partial x^2}\left[D\left(\frac{\partial^2 w}{\partial x^2}+\nu\frac{\partial^2 w}{\partial y^2}\right)\right]-2\frac{\partial^2}{\partial x\partial y}\left[D(1-\nu)\frac{\partial^2 w}{\partial x\partial y}\right]-\\
&-\frac{\partial^2}{\partial y^2}\left[D\left(\frac{\partial^2 w}{\partial y^2}+\nu\frac{\partial^2 w}{\partial x^2}\right)\right]=-p(x,y)
\end{aligned}\tag{5}$$

or

$$\begin{aligned}
&\frac{\partial^2}{\partial x^2}\left(D\frac{\partial^2 w}{\partial x^2}\right)+2\frac{\partial^2}{\partial x\partial y}\left(D\frac{\partial^2 w}{\partial x\partial y}\right)+\frac{\partial^2}{\partial y^2}\left(D\frac{\partial^2 w}{\partial y^2}\right)+\\
&+\nu\left[\frac{\partial^2}{\partial x^2}\left(D\frac{\partial^2 w}{\partial y^2}\right)-2\frac{\partial^2}{\partial x\partial y}\left(D\frac{\partial^2 w}{\partial x\partial y}\right)+\frac{\partial^2}{\partial y^2}\left(D\frac{\partial^2 w}{\partial x^2}\right)\right]=p(x,y)
\end{aligned}\tag{6}$$

The differential equation of the deformed middle surface for plates with variable thickness was obtained.

In equation (6) the differentiation operations are performed and the following notations are considered:

$$D'=\frac{\partial D}{\partial x};\quad D^*=\frac{\partial D}{\partial y};\quad D''=\frac{\partial^2 D}{\partial x^2};\quad D^{**}=\frac{\partial^2 D}{\partial y^2};\quad D'^*=\frac{\partial^2 D}{\partial x\partial y}\tag{7}$$

The deformed middle surface equation can be written as [2]:

INTERSECTII
INTERSECTIONS
http://www.ce.tuiasi.ro/intersections

$$D\Delta\Delta w + 2\left(D'\frac{\partial}{\partial x} + D^{*}\frac{\partial}{\partial y}\right)\Delta w + \left(D'' + D^{**}\right)\Delta w -$$
$$-\left(1-\nu\right)\left(D''\frac{\partial^2 w}{\partial y^2} + 2D'^{*}\frac{\partial^2 w}{\partial x \partial y} + D^{**}\frac{\partial^2 w}{\partial x^2}\right) = p\left(x,y\right) \tag{8}$$

When the thickness variation is only along one direction, as an example y, equation (8) becomes:

$$D\Delta\Delta w + 2D^{*}\frac{\partial}{\partial y}\Delta w + D^{**}\Delta w - \left(1-\nu\right)D^{**}\frac{\partial^2 w}{\partial x^2} = p\left(x,y\right) \tag{9}$$

The few analytical solutions of equation (8) and (9), respectively have been mentioned in introduction and they are referring to particular support and loading cases.

The most frequent solutions are the approximate ones, based on numerical methods (finite difference method, finite element method, boundary element method etc.)

3.THE EFFECT OF THICKNESS VARIATION ON SHEAR FORCES

The shear forces are expressed in terms of twisting and bending moments derivatives, according to relations (3a) and (3b):

$$Q_x = \frac{\partial M_x}{\partial x} + \frac{\partial M_t}{\partial y}; \quad Q_y = \frac{\partial M_y}{\partial y} + \frac{\partial M_t}{\partial x} \tag{10}$$

These relations include the thickness variation, so that, substituting the moments M_x, M_y, M_t by relations (1), they must take into account the rigidity D variation. In consequence, the two shear forces will be:

$$Q_x = -\frac{\partial}{\partial x}D\left(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right) - \frac{\partial}{\partial y}D\left(1-\nu\right)\frac{\partial^2 w}{\partial x \partial y}$$
$$Q_y = -\frac{\partial}{\partial y}D\left(\frac{\partial^2 w}{\partial y^2} + \nu\frac{\partial^2 w}{\partial x^2}\right) - \frac{\partial}{\partial x}D\left(1-\nu\right)\frac{\partial^2 w}{\partial x \partial y} \tag{11}$$

The shear force vector $\{Q_x\ Q_y\}^T$ can be expressed under the following shape:

ISSN 1582-3024

$$\begin{Bmatrix} Q_x \\ Q_y \end{Bmatrix} = -D\begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \end{bmatrix}\begin{Bmatrix} \nabla^2 w \\ \nabla^2 w \end{Bmatrix} + \frac{1}{D}\begin{bmatrix} \frac{\partial D}{\partial x} & 0 & \frac{\partial D}{\partial y} \\ 0 & \frac{\partial D}{\partial y} & \frac{\partial D}{\partial x} \end{bmatrix}\begin{Bmatrix} M_x \\ M_y \\ M_t \end{Bmatrix} \tag{12}$$

and respectively:

$$\begin{Bmatrix} Q_x \\ Q_y \end{Bmatrix} = -D\begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \end{bmatrix}\begin{Bmatrix} \nabla^2 w \\ \nabla^2 w \end{Bmatrix} + \frac{3}{h}\begin{bmatrix} \frac{\partial h}{\partial x} & 0 & \frac{\partial h}{\partial y} \\ 0 & \frac{\partial h}{\partial y} & \frac{\partial h}{\partial x} \end{bmatrix}\begin{Bmatrix} M_x \\ M_y \\ M_t \end{Bmatrix} \tag{13}$$

The second term represents the influence of thickness variation on the shear force. In the particular case, when the derivatives $\partial h / \partial x$ and $\partial h / \partial y$ are constant, the shear stresses τ_{xz}^M and τ_{yz}^M have a linear variation over the plate thickness. In case of a more refined analysis, the energy generated by the shear stresses on corresponding shear strains can be introduced in the elastic strain energy stored by the element.

4. PHYSICAL INTERPRETATION OF SHEAR FORCE CORRECTIONS DUE TO THICKNESS VARIATION

A portion of a plate with continuous variable thickness and two sections parallel to the coordinate planes *x0z* and *y0z* are considered. The plate thickness has the direction of *0z* axis in the local coordinate system and intersects the intrados and extrados surfaces at point a and b, respectively. The bending moments, which produce the tension of intrados fibers and compress the extrados ones, are considered as positive (positive twisting moments are correspondingly introduced according to stresses sign convention).

In the considered zone it is presumed that no surface forces exist, so that, the stresses at the extreme points will be directed along the tangents to the surface lines (Figure 2) [4].

The shear forces have two components, *Q'* and *Q''*, according to relation (13):

$$\begin{Bmatrix} Q_x \\ Q_y \end{Bmatrix} = \begin{Bmatrix} Q'_x + Q''_x \\ Q'_y + Q''_y \end{Bmatrix} \tag{14}$$

The shear forces Q'_x and Q'_y are given by the stresses τ'_{xz}, and τ'_{yz} parabolic distributed over the thickness, and having the maximum values at the middle surface level. The shear forces Q''_x and Q''_y are determined by the shear stresses resulted as a consequence of thickness variation, that is:

$$\tau''_{xz} = \tau''_{xz}(M_x) + \tau''_{xz}(M_t); \quad \tau''_{yz} = \tau''_{yz}(M_y) + \tau''_{yz}(M_t) \tag{15}$$

The point a, located on the intrados surface, at the extremity of the intersection line between the sections parallel to $x0z$ plane and $y0z$ plane, respectively and at this point the section that has $0x$ axis as normal, are considered. The intersection line $\overline{ab}$ marks the limits of the corresponding plate thickness, $h = \overline{ab}$.

The stresses at point a, on the section that has as outward normal $0x$ axis are:

$$\sigma_{xa} = \sigma_{x\max} = \frac{6M_x}{h^2}; \quad \tau_{xya} = \tau_{xy\max} = \frac{6M_t}{h^2} \tag{16}$$

The projections of stresses at point a, tangent to the contour in $0z$ axis direction, which is in fact the direction of thickness h, are:

$$\begin{aligned} &\tau''_{xza}(M_x) + \tau''_{xza}(M_t) = \sigma_{xa} tg\alpha_{xa} + \tau_{xya} tg\alpha_{ya} = \\ &= \sigma_{x\max}\left(\frac{\partial h}{\partial x}\right)_a + \tau_{xy\max}\left(\frac{\partial h}{\partial y}\right)_a \end{aligned} \tag{17}$$

At a point located at a distance z from the middle surface, τ''_{xz} have the expression:

$$\tau''_{xz} = \sigma_{x\max}\frac{2z}{h}\frac{\partial h}{\partial x} + \tau_{xy\max}\frac{2z}{h}\frac{\partial h}{\partial y} \tag{18}$$

By adding these stresses over the plate thickness, the shear force Q''_x is obtained:

$$Q''_x = \int_0^{h/2} \tau''_{xz} dz + \int_0^{-h/2} \tau''_{xz} dz = \frac{3}{h}\left(M_x \frac{\partial h}{\partial x} + M_t \frac{\partial h}{\partial y}\right) \tag{19}$$

that is identical to those resulted in relation (13).

In Fig.2 a, b there are presented the distributions of stresses $\tau''_{xz}(M_x)$, $\tau''_{xz}(M_t)$ that follow a linear variation (considering $\partial h / \partial x$ and $\partial h / \partial y$ as being constant) and in Fig. 2 c the distribution of stresses $\tau'_{xz}(Q_x)$, that have a parabolic variation.

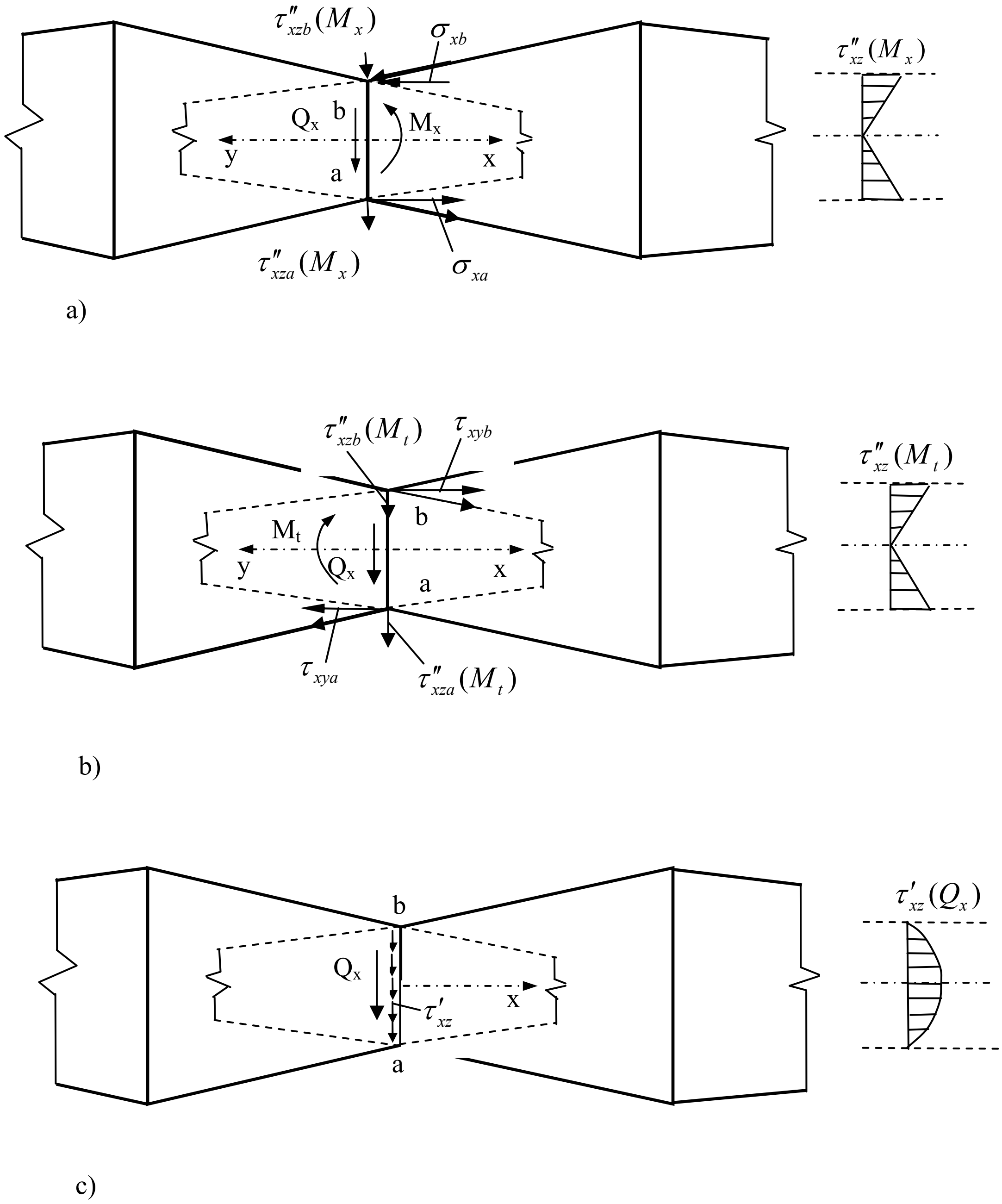

Figure 2. The shear stresses generated by M_x, M_t, Q_x

5. ENERGY EFFECTS

The effect of thickness variation can be energetically pointed out, by including it in the total strain energy of the shear forces Q_x and Q_y. For this reason, the vector of deformations in the local coordinate system, attached to a point of the middle surface, is written as:

$$\{\varepsilon\} = \begin{Bmatrix} \varepsilon_t \\ \varepsilon_n \end{Bmatrix} = \begin{Bmatrix} \varepsilon_M \\ \varepsilon_Q \end{Bmatrix} = \{\{\varepsilon_x, \varepsilon_y, \gamma_{xy}\}; \{\gamma_{xz}, \gamma_{yz}\}\}^T \tag{20}$$

where $\{\varepsilon_t\} = \{\varepsilon_M\} = \{\varepsilon_x, \varepsilon_y, \gamma_{xy}\}^T$ are the deformations in the plane tangent to the middle surface (deformations produced by M_x, M_y, M_t), and $\{\varepsilon_n\} = \{\varepsilon_Q\} = \{\gamma_{xz}, \gamma_{yz}\}^T$ are deformations caused by transverse internal forces, as the shear forces Q_x and Q_y (corrected due to thickness variation).

Similarly the stress vector is divided, so that the constitutive equations in the local coordinate system are:

$$\{\sigma\} = \begin{Bmatrix} \sigma_M \\ \sigma_Q \end{Bmatrix} = \{\{\sigma_x \sigma_y \tau_{xy}\}, \{\tau_{xz} \tau_{yz}\}\}^T = [D]\{\varepsilon\} = \begin{bmatrix} [D_M] & 0 \\ 0 & [D_Q] \end{bmatrix} \begin{Bmatrix} \varepsilon_M \\ \varepsilon_Q \end{Bmatrix} \tag{21}$$

where *[D]* is the material constitutive matrix.

For isotropic materials $[D_M]$ and $[D_Q]$ are:

$$[D_M] = \begin{bmatrix} \bar{\lambda} + 2G & \bar{\lambda} & 0 \\ \bar{\lambda} & \bar{\lambda} + 2G & 0 \\ 0 & 0 & G \end{bmatrix}; \quad [D_Q] = \begin{bmatrix} k_Q G & 0 \\ 0 & k_Q G \end{bmatrix} \tag{22}$$

where $$\bar{\lambda} = \frac{E\nu}{1-\nu^2} \tag{23}$$

is the reduced Lame’s constant for the plane stress state;

E, G, ν – the material elastic constants;

k_Q – correction coefficient due to the non-uniform distribution of transverse shear stresses in *x0z* and *y0z* planes (for homogeneous materials $k_Q = 5/6$).

In the local coordinate system, the total strain energy of the plate can be expressed as:

Structural Mechanics

INTERSECTII
INTERSECTIONS
http://www.ce.tuiasi.ro/intersections

$$\Pi = \frac{1}{2}\int_V \{\varepsilon_M\}^T [D_M] \{\varepsilon_M\} dV + \frac{1}{2}\int_V \{\varepsilon_Q\}^T [D_Q] \{\varepsilon_Q\} dV + U_e \tag{24}$$

where U_e is the position energy of the applied loads.

6. CONCLUSIONS

Sometimes the plate thickness can be continuous variable over the plate surface or over some of its areas. In a particular case, the thickness can be variable in steps, or due to some stiffening elements. There are cases when equivalent systems (as example, due to geometric orthotropy) become with variable thickness.

It is mentioned that the variable thickness does not modify the plate model, in the case of the present paper, a thin plate. However, the thickness variation for plates subjected to bending produces some effects expressed in the internal forces relations and also in the energetic and variational approaches, frequently used in such analyses.

Among these effects we can mention:

- the plate rigidities are variable because of thickness variation;
- the shear forces, as resultants of shear stresses, are influenced by the bending and twisting moments, when the thickness is continuously variable;
- the thickness variation introduces corrections and additional effects, respectively upon deformations and transverse stresses; in this paper, a geometric interpretation has been given to this effect, also pointing out the effects separation;
- in the energy balance the effect of transverse deformation and the effect of redundant stresses produced by the bending and twisting moments must be introduced; the case studies show that for plates with variable thickness, having slopes lying between *5%* and *10%*, the additional energetic effects and transverse deformations effects increase with *100%* [4].

References

1. Timoshenko, St.P., Woinowski-Krieger, S. *Teoria plăcilor plane şi curbe*, Ed. Tehnică, Bucureşti, 1968 (in Romanian), pp. 194-200.
2. Soare, M.V. *Plăci plane*, Secțiunea VI din "Manual pentru calculul construcțiilor", Ed. Tehnică, Bucureşti, 1977 (in Romanian), pp.1000-1001.
3. Ungureanu, N. *Rezistența materialelor şi teoria elasticității*, I.P. Iaşi, 1988 (in Romanian), pp. 270-284.
4. Marțincu, C. *Modele structurale şi metode de analiză statică şi dinamică a plăcilor curbe de grosime variabilă, cu aplicații la structuri hidrotehnice*, teză de doctorat, U.T. "Gh. Asachi" Iaşi, 1998 (in Romanian).
5. Hinton, E., Owen, D.R.J. *Finite Element Software for Plates and Shells*, Pineridge Press, Swansea, U.K., 1984.

6. Zhao, Z., Chen, W. *New Finite Element Model for Analysis of Kirchhoff Plate*, Int. Journ. For Numerical Methods in Engng., 38, 1995, pp. 1201-1214.
7. Karam, V.S., Telles, J.C.F. *On Boundary Elements for Reissner's Plate Theory*, Engng. Analysis, 5, 1988, pp. 21-27.